Elementary embeddings and correctness

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Kunen's theorem: no nontrivial $j : V \to V$

Kunen's inconsistency theorem can be generalized as follows:

Theorem

Suppose $j : M \to N$ is a nontrivial elementary embedding between models of ZFC with the same ordinals. Then at least one of the following holds:

1. *There is $\alpha \in M$ such that $\langle \alpha, j(\alpha), j^2(\alpha), j^3(\alpha), \ldots \rangle \notin M$.*
2. *There is $\alpha \in M$ such that $j[\alpha] \notin N$.*
3. *Some ordinal is regular in M and singular in N.*

Theorem

Suppose $\kappa \leq \lambda$ are regular. κ is λ-supercompact if and only if there is a λ-closed transitive class M and a nontrivial elementary $j : M \to M$ with critical point κ and $\lambda < j(\kappa)$.

Amenable embeddings

We say $j : M \to N$ is *amenable* if $j[x] \in N$ for each $x \in M$.

Example

Suppose κ is Ramsey. Then there is a transitive $M \subseteq V_\kappa$ of size κ and $j : M \to V_\kappa$.

Example

Suppose κ is Ramsey. Then for each $n < \omega$, there are transitive sets $V_\kappa = M_0 \supseteq \cdots \supseteq M_n$, each of size κ, and nontrivial amenable embeddings $j_{k,k-1} : M_k \to M_{k-1}$ for $1 \leq k \leq n$.

Example

Suppose κ is measurable. Then there are transitive sets $V_\kappa = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$, each of size κ, and nontrivial amenable embeddings $j_{k,k-1} : M_k \to M_{k-1}$ for $1 \leq k < \omega$.

Amenable embeddings: fixed point behavior

Example

Suppose κ is measurable. Then there is $M \subsetneq V_\kappa$ such that for each $\delta \in \kappa + 1$, there is $j : M \to V_\kappa$ such that j has exactly ordertype-δ many fixed points above crit(j). They are all inaccessible.

Example

Suppose κ is a measurable limit of measurables. Then there is $M \subsetneq V_\kappa$ such that for each $\delta \in \kappa + 1$ and each $f : \delta \to 2$, there is $j : M \to V_\kappa$ such that j has exactly ordertype-δ many fixed points above crit(j), and for all $i < \delta$, the i^{th} fixed point is regular iff $f(i) = 1$.

Stationary tower

Definition (Woodin)

The *stationary tower forcing* $\mathbb{P}_{<\kappa}$ is the collection of stationary sets that are members of V_κ, ordered by $a \leq b$ if $\bigcup a \supseteq \bigcup b$ and $\{x \cap \bigcup b : x \in a\} \subseteq b$.

Theorem (Woodin)

Suppose κ is an inaccessible limit of completely Jónsson cardinals, and $G \subseteq \mathbb{P}_{<\kappa}$ is generic over V. Let $j : V \to M$ be the direct limit ultrapower embedding of V by G. Then $\kappa \in \mathrm{wfp}(M)$, $j(\kappa) = \kappa$, and $j \restriction V_\kappa$ is amenable.

Stationary tower

Corollary

If there is (a transitive model of) a measurable, then there is a sequence of countable transitive models of ZFC and embeddings:

$$M_0 \xrightarrow{j_0} M_1 \xrightarrow{j_1} M_2 \xrightarrow{j_2} \cdots$$

such that all M_n have the same ordinals, and all j_n are amenable.

What happens for the direct limit M_ω? By proceeding via an iterated forcing defined in M_0, we seem to get either:

1. M_ω is not well-founded.
2. M_ω is well-founded, but has an ordinal which is regular in M_ω and singular in M_0, so the maps $j_{n,\omega}$ are not amenable.

Treating the models like dirt

Let $\delta = M_0 \cap \text{Ord}$. Let $\langle \alpha_i : i < \omega \rangle$ be cofinal in δ. We select our sequence of generics for the stationary towers such that:

1. $\langle \text{crit}(j_{n,n+1}) : n < \omega \rangle$ is increasing and cofinal in δ.
2. For all n and all $i \leq n$, $\text{crit}(j_{n,n+1}) > j_{i,n}(\alpha_n)$.

This implies:

1. The direct limit $M_\omega = \bigcup_{n<\omega} M_n$.
2. For each n and $x \in M_n$, there is $m \geq n$ such that $\text{crit}(j_{m,\omega}) > |x|$, so that $j_{n,\omega}[x] = j_{m,\omega}(j_{n,m}[x]) \in M_\omega$.

By an induction, we can build chains of length any countable ordinal.

Category description

Consider a category consisting of models of ZFC with the same ordinals, and arrows some collection of elementary embeddings between them closed under composition.

For example, given $M \models \text{ZFC}$, let **AmOut**(M) be the category of N with the same ordinals such that there is an amenable $j : M \to N$, with arrows all amenable embeddings between these models.

A partial order can be represented as a category in which there is at most one arrow between any two objects. Besides countable ordinals, what kinds of partial orders appear as subcategories of **AmOut**(M) for some ctm M?

One thing for free: countable well-founded posets.

Incompatibility-preserving embeddings of trees

Theorem (Habič et al.)

If M is a countable transitive model of ZFC and κ is a regular cardinal in M, then there are $\langle G_i : i < \omega \rangle$ which are $\mathrm{Add}(\kappa)$*-generic over M and such that no ZFC model with the same ordinals as M contains any two of them.*

Fact

If $M \models \kappa^{<\kappa} = \kappa$, then for large enough $\theta > \kappa$, the stationary tower $\mathbb{P}_{<\theta}$ absorbs a generic for $\mathrm{Add}(\kappa)$.

So starting with a ctm $M = M_\emptyset$ with enough large cardinals inside, we can build a tree of models $\langle M_\sigma : \sigma \in {}^{<\omega}\omega \rangle$ such that if $\sigma \lhd \tau$, then there is an amenable $j_{\sigma,\tau} : M_\sigma \to M_\tau$, and if $\sigma \perp \tau$, then there is no ZF model $N \supseteq M_\sigma \cup M_\tau$ with the same ordinals.

Incompatibility-preserving embeddings of trees

If we control the critical points like before, we can ensure that direct limit along any branch r yields an amenable $j : M_\sigma \to M_r$ for $\sigma \lhd r$, and M_r has the same ordinals as M.

As before, we can continue to get copies of the tree ${}^{<\alpha}\omega$ for any countable α. So for many ctm M and any countable tree T, there is a "nice" functor $F : T \to \mathbf{AmOut}(M)$, where x, y incomparable implies $F(x), F(y)$ non-amalgamable.

What else can we find in $\mathbf{AmOut}(M)$?

Pseudotrees

Definition

A *pseudotree* is a poset that is linearly ordered below any element.

Let us consider the following pseudotree $T_{\mathbb{Q}}$. Collections of partial functions $f : \mathbb{Q} \to \omega$ such that:

1. $\mathrm{dom}(f)$ is an initial segment of $\mathbb{Q}$ (possibly $\emptyset$).
2. If $\mathrm{dom}(f) \neq \emptyset$, then there is a finite sequence $-\infty = q_0 < q_1 < \cdots < q_n = \max(\mathrm{dom}\, f)$ such that $f \restriction (q_i, q_{i+1}]$ is constant for $i < n$.

Put $f \leq g$ when $f \subseteq g$.

Pseudotrees with meets

Notice that $T_{\mathbb{Q}}$ satisfies the following axiom set (DPT):

1. It's a pseudotree with a least element 0.
2. Every two elements f, g have an infimum $f \wedge g$.
3. Infinite Splitting: For all $g, f_0, \ldots, f_{n-1}$ such that $f_i \wedge f_j = g$ for $i, j < n$, there is a different $f_n > g$ such that $f_i \wedge f_j = g$ for $i, j \leq n$.
4. Density: If $g < f$, there is h such that $g < h < f$.

Lemma

Suppose T satisfies the DPT axioms. Suppose S is a structure in the language $\{\leq, \wedge\}$ satisfying (1) and (2). Suppose S_0 is a finite substructure of S, $\pi : S_0 \to T$ is an embedding, and $x \in S$. Then there is a finite substructure $S_1 \supseteq S_0 \cup \{x\}$ and π can be extended to an embedding $\pi' : S_1 \to T$.

Pseudotrees with meets

Corollary

Any countable pseudotree with meets can be embedded into $T_{\mathbb{Q}}$. Any countable structure satisfying the above axioms is isomorphic to $T_{\mathbb{Q}}$.

Lemma

Any pseudotree is a substructure of a pseudotree with meets.

Proof: Represent the poset as a system of sets under $\subseteq$ and add finite intersections.

Ill-founded magic

Recall: If there is a ctm of ZFC+"There is an inaccessible limit κ of completely Jónsson cardinals," then letting M be the cut of this model at such κ, then for every countable ordinal α, there is a subcategory of **AmOut**(M) isomorphic to ${}^{<\alpha}\omega$, where branching results in non-amalgamability.

If there is (a transitive model of) a measurable, then there are ctm's $M_0 \in M_1$ such that $M_1 \models M_0$ is a ctm of ZFC + "There is inaccessible limit of completely Jónsson cardinals." We can take M_1 to be of the form $L_\xi[r]$ for some real r and countable ordinal ξ.

The above reasoning shows that $L_\xi[r] \models$ "For all countable α, there is a nice functor from ${}^{<\alpha}\omega$ to **AmOut**(M), for appropriate $M \in M_0$.

Ill-founded magic

Fact

If we force with $\mathcal{P}(\omega_1)/\mathrm{NS}$ over $L_\xi[r]$, then we get a generic embedding $j : L_\xi[r] \to U$ such that ω_1^U is ill-founded, but $\omega_2^{L_\xi[r]} \subseteq \mathrm{wfp}(U)$.

In particular, $j(M_0) = M_0$.

Let δ be the set of well-founded ordinals in Ord^U. For each $\gamma \in \mathrm{Ord}^U$, U has a decomposition by Cantor Normal Form,

$$\alpha = \omega^{\beta_n} \cdot k_n + \cdots + \omega^{\beta_m} \cdot k_m + \omega^{\beta_{m-1}} \cdot k_{m-1} + \cdots + \omega^{\beta_0} \cdot k_0,$$

Where $\beta_n > \cdots > \beta_0$, $\beta_0, \ldots, \beta_{m-1} < \delta$ and $\beta_m, \ldots, \beta_n$ are ill-founded.

Let α be an ill-founded ordinal that U believes is countable. U believes that there is a nice functor from ${}^{<\alpha}\omega$ to $\mathbf{AmOut}(M)$.

The collection of levels of ${}^{<\alpha}\omega$ which have trivial well-founded component is a dense linear order. The restriction of ${}^{<\alpha}\omega$ to these levels satisfies DPT.

Conclusions and Questions

In particular, every countable linear order is isomorphic to a subcategory of **AmOut**(M). For cheap we get the same for countable partial orders.

We can also show that this category does not contain a subcategory isomorphic to an uncountable linear order.

Question

Is it possible to find M_0, M_1, N models of ZFC with the same ordinals, and M_0, M_1 are $\subseteq$-incomparable, and there are (amenable) $j_0 : M_0 \to N$, $j_1 : M_1 \to N$?

Question

Is it possible to find $M \neq N$ models of ZFC with the same ordinals such that there are elementary $j : M \to N$ and $i : N \to M$?